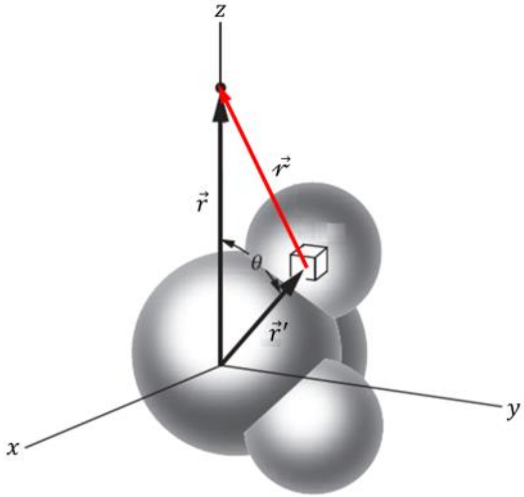
—Chapter 10—

Electric Fields in Matter

10-1 The Moment of Charge Distribution

A. THE MOMENT OF CHARGE DISTRIBUTION

(1) The potential of any charge distribution at $\vec{r}$ is given by



$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{\mathscr{r}} d\tau'$$

where $\mathscr{r} = \left|\vec{r} - \vec{r}'\right|$, the distance from the charge element to the point $\mathscr{r}$. Write $1/\mathscr{r}$ in the form of a power series with Legendre polynomials,

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$$= \frac{1}{r} + \frac{r'}{r^2}\cos\theta + \frac{1}{r^3}r'^2\frac{\left(3\cos^2\theta - 1\right)}{2} + \cdots$$

where θ is the angle between $\vec{r}$ and $\vec{r}'$. Thus, we obtain

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{r} + \frac{r'}{r^2}\cos\theta + \frac{1}{r^3}r'^2\frac{\left(3\cos^2\theta - 1\right)}{2}\right]\rho(r')\,d\tau'$$
$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(r')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta\,\rho(r')\,d\tau' + \frac{1}{2r^3}\int r'^2\left(3\cos^2\theta - 1\right)\rho(r')\,d\tau' + \cdots\right]$$

The integral depends only on the charge distribution.

EXAMPLES:

1. Consider a spherical shell with uniform surface charge density σ. Find the potential at $r > R$?

   ANSWER:

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(r')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta\,\rho(r')\,d\tau' + \frac{1}{2r^3}\int r'^2\left(3\cos^2\theta - 1\right)\rho(r')\,d\tau' + \cdots\right]$$

$$\int\rho(r')\,d\tau' = \int\sigma\,R^2\sin\theta\,d\theta d\phi = \sigma 4\pi R^2$$

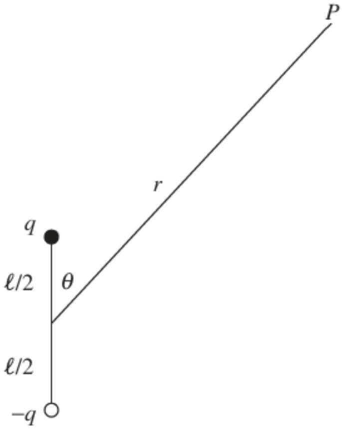
$$\int r'\cos\theta\,\rho(r')\,d\tau' = \int R\cos\theta\,\sigma\,R^2\sin\theta\,d\theta\int d\phi$$
$$= \sigma R^3\int_0^\pi\cos\theta\,d(-\cos\theta)\cdot 2\pi$$
$$= -\sigma 2\pi R^3\left.\frac{\cos^2\theta}{2}\right|_0^\pi$$
$$= 0$$

$$\int r'^2\left(3\cos^2\theta - 1\right)\rho(r')\,d\tau' = \int\sigma R^2\left(3\cos^2\theta - 1\right)R^2\sin\theta\,d\theta d\phi$$
$$= \sigma 2\pi R^4\int\left(3\cos^2\theta - 1\right)d(-\cos\theta)$$
$$= \sigma 2\pi R^4\left.\left(-\cos^3\theta + \cos\theta\right)\right|_0^\pi$$
$$= 0$$

Therefore, its only nonzero term is the leading term, which is also called the monopole. Thus, we obtain

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(r')\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sigma 4\pi R^2 = \frac{\sigma}{\epsilon_0}\frac{R^2}{r}$$

2. Consider a setup with two equal and opposite charges $\pm q$ located at positions $z = \pm l/2$ on the z-axis. Find the potential at $r > l$?



ANSWER:

Since θ is fixed, $\cos\theta$ can be took outside the integral.

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(r')\,d\tau' + \frac{\cos\theta}{r^2}\int r'\rho(r')\,d\tau' + \frac{(3\cos^2\theta - 1)}{2r^3}\int r'^2\rho(r')\,d\tau' + \cdots\right]$$

Since

$$\int \rho(r')\,d\tau' = q - q = 0$$

$$\int r'\rho(r')\,d\tau' = \left(q\frac{l}{2} + (-q)\left(-\frac{l}{2}\right)\right) = ql$$

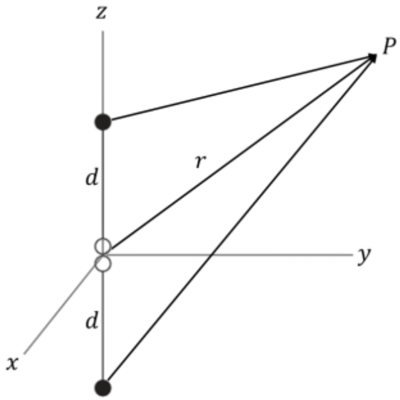
$$\int r'^2\rho(r')\,d\tau' = \left(q\left(\frac{l}{2}\right)^2 + (-q)\left(-\frac{l}{2}\right)^2\right) = 0$$

therefore, its only nonzero term is the second term, which is also called the dipole. Thus, we obtain

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\int r'\rho(r')\,d\tau' = \frac{ql}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

3. Consider a setup with four equal and opposite charges $\pm q$ located

at positions $z = 0, \pm d/\sqrt{2}$ on the z-axis. Find the potential at $r > d$?



ANSWER:

Since θ is fixed, $\cos\theta$ can be took outside the integral.

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(r')\,d\tau' + \frac{\cos\theta}{r^2}\int r'\rho(r')\,d\tau' + \frac{(3\cos^2\theta - 1)}{2r^3}\int r'^2\rho(r')\,d\tau' + \cdots\right]$$

Since

$$\int \rho(r')\,d\tau' = q - 2q + q = 0$$

$$\int r'\rho(r')\,d\tau' = qd + (-2q)0 + q(-d) = 0$$

$$\int r'^2\rho(r')\,d\tau' = qd^2 + (-2q)0 + q(-d)^2 = 2qd^2$$

therefore, its only nonzero term is the third term, which is also called the linear quadrupole. Thus, we obtain

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{(3\cos^2\theta - 1)}{2r^3}\int r'^2\rho(r')\,d\tau' = \frac{2qd^2}{4\pi\epsilon_0}\frac{(3\cos^2\theta - 1)}{2r^3}$$

(2) This power series is called the multipole expansion of the potential.

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\Bigg[\underbrace{\frac{1}{r}\int \rho(r')\,d\tau'}_{\substack{\text{monopole}\\ \propto 1/r}} + \underbrace{\frac{1}{r^2}\int r'\cos\theta\,\rho(r')\,d\tau'}_{\substack{\text{dipole}\\ \propto 1/r^2}} + \underbrace{\frac{1}{2r^3}\int r'^2\left(3\cos^2\theta - 1\right)\rho(r')\,d\tau'}_{\substack{\text{quadrupole}\\ \propto 1/r^3}} + \cdots\Bigg]$$

Since

$$\int \rho(r')\,d\tau' = q$$

$$\int r'\cos\theta\,\rho(r')\,d\tau' = \int \hat{r}\cdot\vec{r}'\rho(r')\,d\tau' = \hat{r}\cdot\int \vec{r}'\rho(r')\,d\tau'$$

$$\begin{aligned}\int r'^2\left(3\cos^2\theta - 1\right)\rho(r')\,d\tau' &= \int\left(3\left(\hat{r}\cdot\vec{r}'\right)^2 - |\hat{r}|^2\left|\vec{r}'\right|^2\right)\rho(r')\,d\tau' \\ &= \int\left(3\left(\hat{r}^\dagger\cdot\vec{r}'\right)\left(\vec{r}'^\dagger\cdot\hat{r}\right) - \left|\hat{r}^\dagger\cdot\hat{r}\right|\left|\vec{r}'\right|^2\right)\rho(r')\,d\tau' \\ &= \hat{r}^\dagger\cdot\int\left(3\vec{r}'\vec{r}'^\dagger - \left|\vec{r}'\right|^2\right)\rho(r')\,d\tau'\cdot\hat{r}\end{aligned}$$

we then define

$$\vec{p} = \int \vec{r}'\rho(r')\,d\tau' \cdots\cdots \text{called dipole moment}$$

$$\mathbf{Q} = \int\left(3\vec{r}'\vec{r}'^\dagger - |r'|^2\right)\rho(r')\,d\tau' \cdots\cdots \text{called quadrupole moment}$$

and get

$$\int r'\cos\theta\,\rho(r')\,d\tau' = \vec{p}\cdot\hat{r}$$

$$\int r'^2\left(3\cos^2\theta - 1\right)\rho(r')\,d\tau' = \hat{r}^\dagger\cdot\mathbf{Q}\hat{r}$$

Thus, we obtain

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\Bigg[\underbrace{\frac{q}{r}}_{\text{monopole}} + \underbrace{\frac{\vec{p}\cdot\hat{r}}{r^2}}_{\text{dipole}} + \underbrace{\frac{\hat{r}^\dagger\cdot\mathbf{Q}\hat{r}}{2r^3}}_{\text{quadrupole}} + \cdots\Bigg]$$

where the quantities q, $\vec{p}$, and $\mathbf{Q}$ depend only on the distribution, and the dipole moment vector points in the direction from the negative charge to the positive charge.

B. THE ELECTRIC FIELD OF THE MOMENT

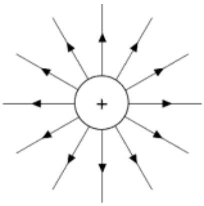
(1) Monopole moment

The potential of a monopole is given by

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \text{where } q = \int \rho(r')\, d\tau'$$

which is spherical symmetry.

The electric field is



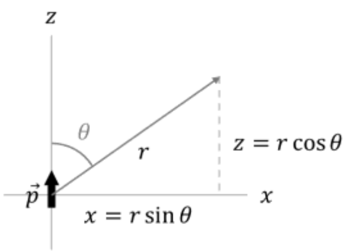
$$\vec{E} = -\nabla\varphi = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

(2) Dipole moment

The potential of a dipole is given by

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{where } \vec{p} = \int \vec{r}'\rho(r')\, d\tau' = \text{dipole moment}$$

If the dipole is located at the origin and pointed in the z-direction, the potential is symmetrical around the z-axis.



Thus, we have

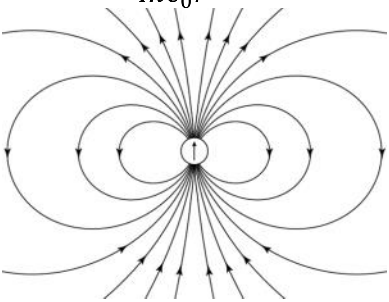
$$\varphi(r) = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}}_{\substack{\text{spherical}\\ \text{coordinates}}} = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{pz}{(x^2+z^2)^{3/2}}}_{\substack{\text{Cartesian}\\ \text{coordinates}}}$$

The electric field in Cartesian coordinates is

$$\begin{aligned}\vec{E} &= -\nabla\varphi \\ &= -\frac{\partial\varphi}{\partial x}\hat{x} - \frac{\partial\varphi}{\partial y}\hat{y} - \frac{\partial\varphi}{\partial z}\hat{z} \\ &= \frac{p}{4\pi\epsilon_0}\frac{3zx}{(x^2+z^2)^{5/2}}\hat{x} + \frac{p}{4\pi\epsilon_0}\left(\frac{3z^2}{(x^2+z^2)^{5/2}} - \frac{1}{(x^2+z^2)^{3/2}}\right)\hat{z}\end{aligned}$$

The electric field in spherical coordinates is

$$\begin{aligned}\vec{E} &= -\nabla\varphi \\ &= -\frac{\partial\varphi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\hat{\phi} \\ &= \frac{p}{4\pi\epsilon_0}\frac{2\cos\theta}{r^3}\hat{r} + \frac{1}{r}\frac{p}{4\pi\epsilon_0}\frac{\sin\theta}{r^2}\hat{\theta} \\ &= \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)\end{aligned}$$

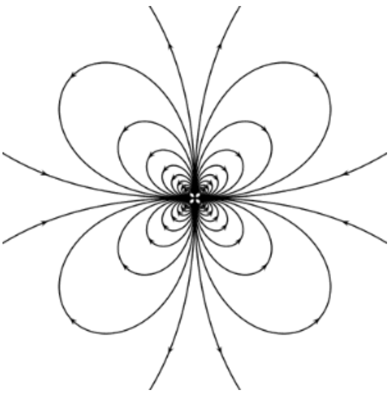


(3) Quadrupole moment

The potential of a quadrupole is given by

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}^\dagger \cdot \mathbf{Q}\hat{r}}{2r^3} \quad \text{where } \mathbf{Q} = \int \left(3\vec{r}'\vec{r}'^\dagger - |r'|^2\right)\rho(r')\,d\tau'$$

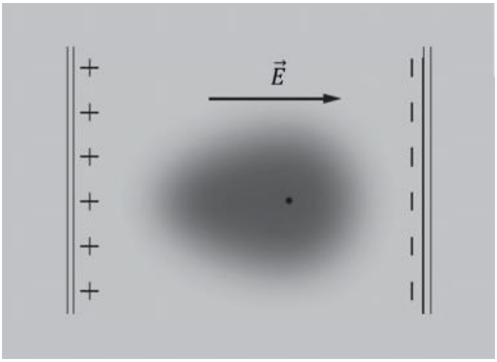
The electric field is



10-2 Polarization

A. INDUCED DIPOLE

(1) For some neutral atoms, when they are placed in the electric field, they will produce a dipole moment. Such an atom is called polarized and a dipole is called *induced dipole*.



Typically, this induced dipole moment is approximately proportional to the field as

$$\vec{p} = \alpha\vec{E}$$

The constant α is called atomic polarizability.

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.67	0.396	24.1	1.64	43.4	59.4

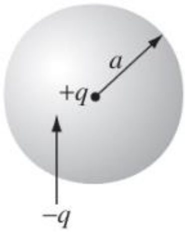
OS:

$$\frac{\alpha}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\ \mathrm{m}^3 \text{ for hydrogen atom}$$

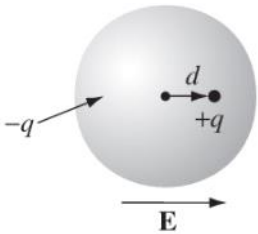
Hand book of Chemistry and Physics, 91st ed. (Boca Raton: CRC Press, 2010).

EXAMPLES:

1. Consider a neutral atom, consisting of a point nucleus $(+q)$ surrounded by a uniformly charged spherical cloud $(-q)$ of radius a,



Now, place the neutral atom in a uniform field $\vec{E}$, the nucleus will be shifted slightly to the right and the electron cloud to the left.



Find the atomic polarizability of a polarized hydrogen atom.

ANSWER:

At equilibrium, the field produced by the electron cloud is

$$\vec{E} = -\vec{E}_{\text{in}}$$

Using Gauss's law, we have

$$\oint_S \vec{E}_{\text{in}} \cdot d\vec{a} = \frac{q_d}{\epsilon_0} \Rightarrow E_{\text{in}} \cdot 4\pi d^2 = \frac{q_d}{\epsilon_0} \Rightarrow E_{\text{in}} = \frac{1}{4\pi\epsilon_0} \frac{q_d}{d^2}$$

Since

$$q_d = \frac{4}{3}\pi d^3 \rho$$

$$q = \frac{4}{3}\pi a^3 \rho$$

we have

$$E_{\text{in}} = \frac{1}{4\pi\epsilon_0} \frac{1}{d^2} \left(\frac{4}{3}\pi d^3\right) \left(\frac{q}{\frac{4}{3}\pi a^3}\right) = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

We let the dipole moment be

$$\vec{p} = q\vec{d}$$

and obtain the field inside the sphere point to the opposite direction of $\vec{E}$, i.e.,

$$\vec{E}_{\text{in}} = -\frac{1}{4\pi\epsilon_0} \frac{\vec{p}}{a^3}$$

The hydrogen atom is polarized by the external field $\vec{E}$ and produces a dipole moment $\vec{p}$.

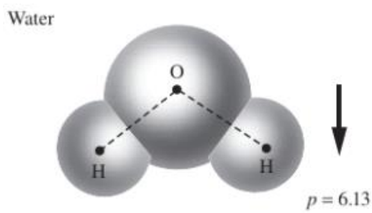
$$\vec{E} = -\vec{E}_{\text{in}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{a^3}$$
$$\vec{p} = 4\pi\epsilon_0 a^3 \vec{E} \Rightarrow \alpha = 4\pi\epsilon_0 a^3$$
Thus, the uniform polarized mode gives atomic polarizability of the polarized hydrogen atom as
$$\frac{\alpha}{4\pi\epsilon_0} = a^3 = \left(0.5 \times 10^{-10}\right)^3 = 0.13 \times 10^{-30}\ \text{m}^3$$

B. PERMANENT DIPOLE

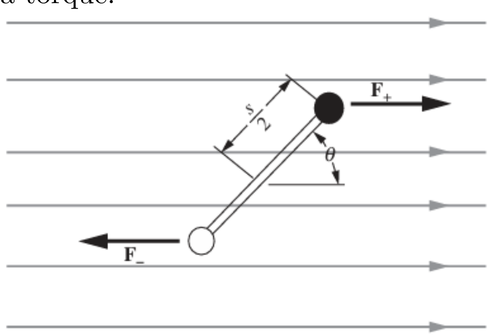
(1) For some molecules, they have built-in electric dipole moment even in the absence of an electric field. Such a dipole is called permanent dipole and the molecule is called *polar molecule*.

EXAMPLES:

1. For the water molecule, the O-H bonds bent in the middle. The bending of the O-H axes makes an angle of about $105°$ with one another. This leaves a negative charge at the vertex and a net positive charge on the opposite side. Thus, the water molecule produces a dipole moment.

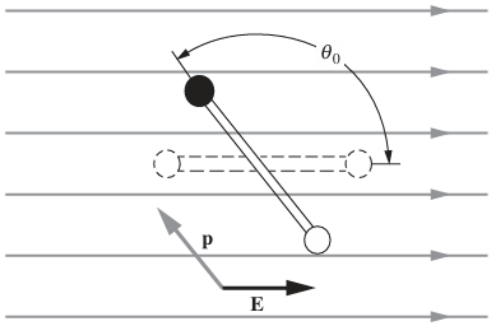


(2) A permanent dipole, in a uniform external field, obviously experiences a torque.



$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{\vec{s}}{2} \times q\vec{E} + \left(-\frac{\vec{s}}{2}\right) \times (-q)\vec{E} = q\vec{s} \times \vec{E} = \vec{p} \times \vec{E}$$

This torque rotates the dipole unless it is placed parallel or anti-parallel to the field.



If we apply an external and opposite torque, it neutralizes the effect of this torque given by $\vec{\tau}$ and it rotates the dipole from the angle θ_0 to an angle θ at an infinitesimal angular speed without any angular acceleration. The amount of work done by the external torque can be given by

$$W = \int_{\theta_0}^{\theta} \vec{\tau} \cdot d\vec{\theta} = \int_{\theta_0}^{\theta} pE \sin\theta \, d\theta = -pE(\cos\theta_0 - \cos\theta)$$

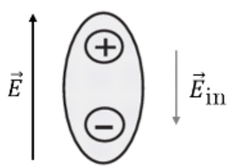
Considering the initial angle to be the angle at which the potential energy is zero, the potential energy of the system can be given as,

$$U = -W = pE\left(\cos\frac{\pi}{2} - \cos\theta\right) = -pE\cos\theta = -\vec{p} \cdot \vec{E}$$

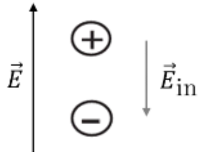
C. POLARIZATION

(1) With an applied field, it is possible to separate the charges in the constituent atoms and molecules.

There can then be an induced dipole moment in the material.



It is also possible that the constituents in the material have a permanent electric dipole moment, which can be lined up in the applied field



In both cases, the induced field arising from the aligned dipoles acts to reduce the applied field.

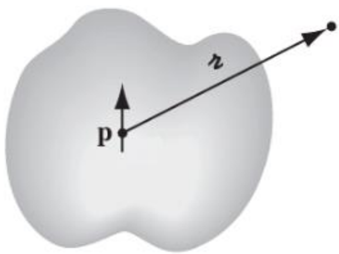
(2) A material is polarized when a lot of little dipoles, induced or built-in, point along the direction of the field. A convenient measure of this effect is

$$\vec{P} = \vec{p}N \equiv \text{dipole moment per unit volume}$$

which is called the **polarization**.

D. THE FIELD PRODUCED BY POLARIZED MATTER

(1) Suppose we have a piece of polarized material - that is, an object containing a lot of microscopic dipoles $\vec{p}$ lined up.



The polarization $\vec{P}$ is given. The electric potential, at some external point, is

$$\varphi = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Since the gradient acts on a function of $\vec{r}-\vec{r}'$ gives [c.f.1-4]

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

thus, we have

$$\varphi = \frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

Using the identity of product rules:

$$\nabla' \cdot \left(\frac{1}{r}\vec{P}\right) = \frac{1}{r}\nabla' \cdot \vec{P} + \vec{P} \cdot \nabla' \frac{1}{r}$$

The potential becomes

$$\varphi = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \nabla' \cdot \left(\frac{\vec{P}}{r}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{r}\left(\nabla' \cdot \vec{P}\right) d\tau'\right]$$

Then, applying Gauss's divergence theorem to the first term, we get

$$\begin{aligned}\varphi &= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\vec{P}}{r} \cdot d\vec{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{r}\left(\nabla' \cdot \vec{P}\right) d\tau' \\ &= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{r}\underbrace{\left(\vec{P} \cdot \hat{n}'\right)}_{\substack{\text{surface}\\\text{charge}\\\text{density}}} da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{r}\underbrace{\left(-\nabla' \cdot \vec{P}\right)}_{\substack{\text{volume}\\\text{charge}\\\text{density}}} d\tau'\end{aligned}$$

We then let

$\sigma_b = \vec{P}\left(\vec{r}'\right) \cdot \hat{n}' = P(r')\cos\theta$ ····· surface charge density

$\rho_b = -\nabla' \cdot \vec{P}\left(\vec{r}'\right)$ ····· volume charge density

σ_b and ρ_b are called bound charge.

(2) The field produced by a polarized matter is

$$\begin{aligned}\vec{E} &= -\nabla\varphi \\ &= -\frac{1}{4\pi\epsilon_0}\nabla\oint_{\mathcal{S}} \frac{\sigma_b(r')}{r} da' - \frac{1}{4\pi\epsilon_0}\nabla\int_{\mathcal{V}} \frac{\rho_b(r')}{r} d\tau' \\ &= -\frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \left(\nabla\frac{1}{r}\right)\sigma_b(r')\, da' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \left(\nabla\frac{1}{r}\right)\rho_b(r')\, d\tau' \\ &= \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\hat{r}}{r^2}\sigma_b\, da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{r}}{r^2}\rho_b\, d\tau'\end{aligned}$$

Consider a case of a uniformly polarized matter, i.e., $\vec{P}$ = constant.
Since

$$\rho_b = -\nabla' \cdot \vec{P} = 0$$

we have

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\hat{r}}{r^2}\sigma_b\, da' = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\hat{r}}{r^2}\vec{P} \cdot \hat{n}' da' = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\hat{r}}{r^2}\vec{P} \cdot d\vec{a}'$$

Using Gauss's divergence theorem, we obtain

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \left(\nabla' \cdot \vec{P}\right)\frac{\hat{r}}{r^2} d\tau' = \vec{P} \cdot \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \nabla' \frac{\hat{r}}{r^2} d\tau'$$

Using the identity:

$$\nabla = -\nabla'$$

we obtain

$$\vec{E} = \vec{P} \cdot \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} (-\nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

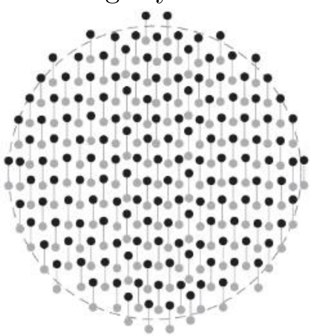
$$= -\vec{P} \cdot \nabla \left(\frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \right)$$

$$= -\left(\vec{P} \cdot \nabla\right) \vec{E}'$$

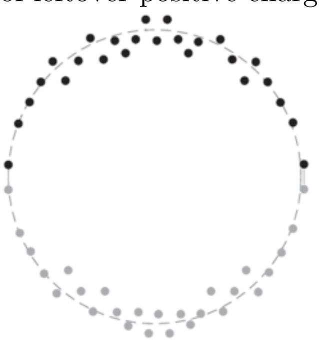
where $\vec{E}'$ is the field produced by the same matter with a uniform charge density $\rho = 1$.

(3) Physical interpretation of $\vec{E} = -\left(\vec{P} \cdot \nabla\right) \vec{E}'$:

Suppose that there is a uniformly polarized sphere. The sphere consists of two spheres of charges: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the sphere is uniformly polarized, all the positive charges move slightly upward (the z direction), and all the negative charges move slightly downward.



The two spheres no longer overlap perfectly: at the top there's a "cap" of leftover positive charge and at the bottom a cap of negative charge.

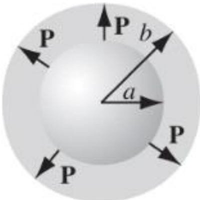


This "leftover" charge is the bound surface charge σ_b.

$$\sigma_b = \vec{P} \cdot \hat{n} = P \cos\theta$$

EXAMPLES:

1. A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization $\vec{P}(r) = \frac{k}{r}\hat{r}$, where k is a constant and r is the distance from the center. Find the electric field.



ANSWER:

$$\rho_b = -\nabla \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2}$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} \frac{k}{b} \quad , & r = b \\ -\frac{k}{a}, & r = a \end{cases}$$

For $r < a$:

$$q_{\text{enc}} = 0$$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0} = 0 \Rightarrow \vec{E} = 0$$

For $a < r < b$:

$$q_{\text{enc}} = \oint_S \sigma_b da + \int_V \rho_b \, d\tau$$

$$= \int -\frac{k}{a} a^2 \sin\theta \, d\theta d\phi + \int_a^r -\frac{k}{r^2} r^2 \sin\theta \, dr d\theta d\phi$$

$$= -ka \cdot 4\pi - k(r-a) \cdot 4\pi$$

$$= -4\pi k r$$

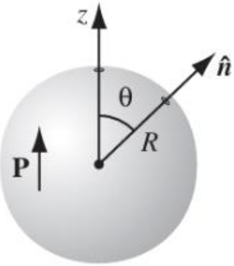
$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0} = -\frac{4\pi k r}{\epsilon_0} \Rightarrow E \cdot 4\pi r^2 = -\frac{4\pi k r}{\epsilon_0} \Rightarrow \vec{E} = -\frac{k}{\epsilon_0 r}\hat{r}$$

For $r > b$:

$$q_{\text{enc}} = 0$$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{q_{\text{enc}}}{\epsilon_0} = 0 \Rightarrow \vec{E} = 0$$

2. Find the electric field produced by a uniformly polarized sphere of radius R.



ANSWER:

- Method I:

Since $\vec{P}$ is uniform, $\nabla \cdot \vec{P} = 0$ and $\sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta$. Thus we need to find the potential by a charge density $P\cos\theta$ plastered over the surface of a sphere.

The potential in spherical coordinates is given by the method of separation of variables [c.f.3-3] as

$$\varphi = \begin{cases} \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \geq R \\ \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \leq R \end{cases}$$

The boundary conditions are

(i) at the surface of the sphere, the potential is continuous

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \Rightarrow A_l R^l = \frac{B_l}{R^{l+1}}$$

(ii) the radial derivative of φ suffers a discontinuity

$$\frac{\partial \varphi_{\text{out}}}{\partial r} - \frac{\partial \varphi_{\text{in}}}{\partial r} = -\frac{\sigma_b}{\epsilon_0}$$

$$\Rightarrow -\sum_{l=0}^{\infty} (l+1) \frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{\sigma_b}{\epsilon_0}$$

$$\Rightarrow \sum_{l=0}^{\infty} (l+1) \frac{A_l R^{2l+1}}{R^{l+2}} P_l(\cos\theta) + \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = \frac{\sigma_b}{\epsilon_0}$$

$$\Rightarrow \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{\sigma_b}{\epsilon_0}$$

Using the orthogonality relation,

$$\frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta = \delta_{l,l'} = \begin{cases}0, & \text{for } l \neq l' \\ 1, & \text{for } l = l'\end{cases}$$

we have

$$\frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta)\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta)\sin\theta\,d\theta$$

$$= \frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta)\frac{\sigma_b}{\epsilon_0}\sin\theta\,d\theta$$

$$\Rightarrow \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta$$

$$= \frac{2l'+1}{2}\int_0^\pi \frac{\sigma_b}{\epsilon_0}P_{l'}(\cos\theta)\sin\theta\,d\theta$$

$$\Rightarrow \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\,\delta_{l,l'} = \frac{2l'+1}{2}\int_0^\pi \frac{\sigma_b}{\epsilon_0}P_{l'}(\cos\theta)\sin\theta\,d\theta$$

Thus, we can obtain A_l as

$$(2l'+1)A_{l'}R^{l'-1} = \frac{2l'+1}{2}\int_0^\pi \frac{\sigma_b}{\epsilon_0}P_{l'}(\cos\theta)\sin\theta\,d\theta$$

$$\Rightarrow A_{l'} = \frac{1}{\epsilon_0 R^{l'-1}}\cdot\frac{1}{2}\int_0^\pi \sigma_b P_{l'}(\cos\theta)\sin\theta\,d\theta$$

$$= \frac{1}{\epsilon_0 R^{l'-1}}\cdot\frac{1}{2}\int_0^\pi P\cos\theta\,P_{l'}(\cos\theta)\sin\theta\,d\theta$$

Since $P_1(\cos\theta) = \cos\theta$, we have

$$A_{l'} = \frac{P}{\epsilon_0 R^{l'-1}(2l'+1)}\cdot\frac{2l'+1}{2}\int_0^\pi P_1(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta$$

$$= \frac{1}{\epsilon_0 R^{l'-1}(2l'+1)}\cdot\delta_{1,l'}$$

$$\Rightarrow A_{l'} = \begin{cases}0 & , \quad \text{for } l' \neq 1 \\ \dfrac{P}{\epsilon_0 R^{1-1}(2\cdot 1+1)} = \dfrac{P}{3\epsilon_0}, & \quad \text{for } l' = 1\end{cases}$$

Therefore, the potential is

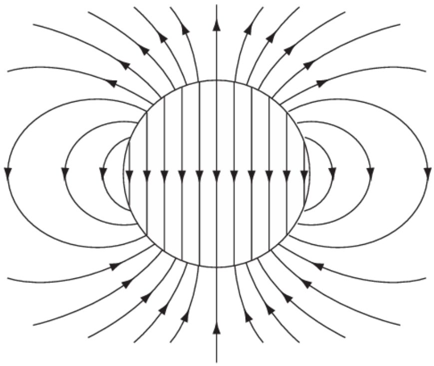
$$\varphi = \begin{cases}\dfrac{B_1}{r^2}P_1(\cos\theta) = \dfrac{A_1 R^{1+2}}{r^2}\cos\theta = \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for } r \geq R \\ A_1 r P_1(\cos\theta) = \dfrac{P}{3\epsilon_0}r\cos\theta & , \text{for } r \leq R\end{cases}$$

The field outside the sphere is

$$
\begin{aligned}
\vec{E} &= -\nabla\varphi \\
&= -\frac{\partial\varphi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\hat{\phi} \\
&= \frac{P}{3\epsilon_0}\frac{2R^3\cos\theta}{r^3}\hat{r} + \frac{P}{3\epsilon_0}\frac{R^3\sin\theta}{r^3}\hat{\theta} \\
&= \frac{R^3 P}{3\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)
\end{aligned}
$$

Inside the sphere, we have $r\cos\theta = z$ and obtain

$$
\vec{E}_{\text{in}} = -\nabla\varphi = -\frac{P}{3\epsilon_0}\hat{z} = -\frac{\vec{P}}{3\epsilon_0}
$$



- Method II:

  Using Gauss's law, the electric field of a sphere with uniform charge density $\rho = 1$ is given as

$$
\vec{E}' = \begin{cases} \dfrac{1}{3\epsilon_0} r\hat{r} = \dfrac{1}{3\epsilon_0}\vec{r} \quad , & r < R \\ \dfrac{1}{4\pi\epsilon_0 r^2}\hat{r} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{r}}{r^3}, & r > R \end{cases}
$$

For $r < R$

$$
\vec{E} = -\left(\vec{P}\cdot\nabla\right)\vec{E}' = -\frac{P}{3\epsilon_0}\frac{\partial}{\partial r}\vec{r} = -\frac{P}{3\epsilon_0}\hat{r} = -\frac{\vec{P}}{3\epsilon_0}
$$

For $r > R$

In Cartesian coordinates:

$$
\vec{E}' = \frac{1}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{\left(x^2 + y^2 + z^2\right)^{3/2}}
$$

$$E_x = -\left(P_x \frac{\partial}{\partial x} + P_y \frac{\partial}{\partial y} + P_z \frac{\partial}{\partial z}\right) \frac{1}{4\pi\epsilon_0} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{P_x}{r^3} - \frac{3x}{r^5}\left(P_x x + P_y y + P_z z\right)\right]$$

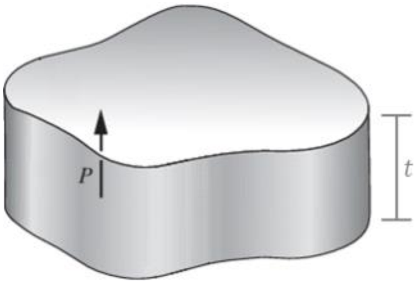
$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{P}}{r^3} - \frac{3\vec{r}\left(\vec{P}\cdot\vec{r}\right)}{r^5}\right]_x$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{P} - 3\left(\vec{P}\cdot\hat{r}\right)\hat{r}}{r^3}\right]_x$$

Thus, we obtain

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{3\left(\vec{P}\cdot\hat{r}\right)\hat{r} - \vec{P}}{r^3}$$

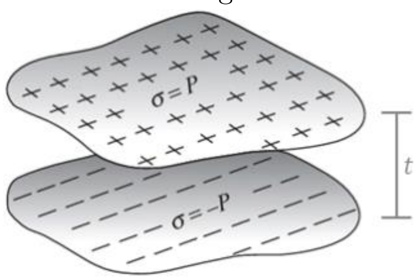
3. Find the electric field inside a uniformly polarized slab.



ANSWER:

$\sigma_b = \vec{P}\cdot\hat{n} = P$

The surface charges are at the top and bottom surface.



The potential between two plates is

$$\varphi = \frac{\sigma t}{\epsilon_0} = \frac{Pt}{\epsilon_0}$$

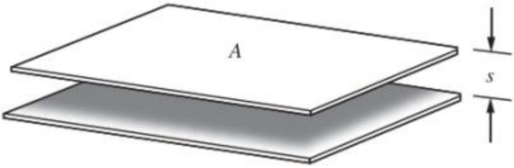
The field inside the slab is

$$\vec{E}_{\mathrm{in}} = -\frac{\varphi}{t} = -\frac{\vec{P}}{\epsilon_0}$$

10-3 Dielectric Constant

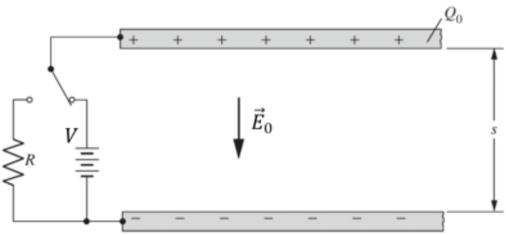
A. DIELECTRICS - POLARIZABLE INSULATORS

(1) For a parallel-plate capacitor, the capacitance is given by



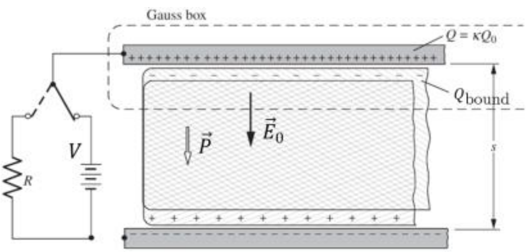
$$C_0 = \frac{Q_0}{V} = \frac{\epsilon_0 A}{s}$$

The field between the two plates is



$$E_0 = \frac{V}{s} = \frac{\sigma}{\epsilon_0} = \frac{Q_0}{\epsilon_0 A} \quad \text{where } Q_0 = \sigma A$$

(2) Suppose we fill the space between the two plates with a slab of insulator. If the material made up the slab is polarized, then it is called a dielectric.



Since V is fixed, $\vec{E}_0$ becomes the total field of the dielectric slab as

$$\vec{E}_0 = \vec{E} + \vec{E}_{\text{in}} = \vec{E} - \frac{\vec{P}}{\epsilon_0}$$

Thus, we use Gauss's law and obtain

$$\oint_S \vec{E}_0 \cdot d\vec{a} = E_0 A = \frac{Q_{\text{enc}}}{\epsilon_0}$$

where

$$Q_{\text{enc}} = Q + Q_{\text{bound}} = Q_0$$

Assume that at fixed voltage, the charge Q on the top conducting plate is proportional to Q_0 when an empty capacitor is filled with a dielectric material,

$Q = \kappa Q_0 \cdots\cdots \kappa$ is called the **dielectric constant**

which also implies that

$$\vec{E} = \frac{Q}{\epsilon_0 A} = \frac{\kappa Q_0}{\epsilon_0 A} = \kappa \vec{E}_0 \cdots\cdots \text{(a)}$$

Then, we obtain

$$\vec{E}_0 = \kappa \vec{E}_0 - \frac{\vec{P}}{\epsilon_0} \Rightarrow \vec{P} = \epsilon_0(\kappa - 1)\vec{E}_0$$

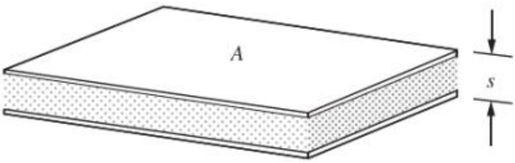
Thus, the field $\vec{E}_0$ in the dielectric causes $\vec{P}$. When $\vec{P}$ is proportional to $\vec{E}_0$, the matter is called linear dielectric.

Sometimes we would express $\vec{P}$ as

$$\vec{P} = \chi_e \epsilon_0 \vec{E}_0$$

where $\chi_e = \kappa - 1$ is called the electric susceptibility of the dielectric matter.

(3) The capacitance of a dielectric is given by



$$C = \frac{Q}{V} = \frac{\kappa Q_0}{V} = \kappa C_0$$

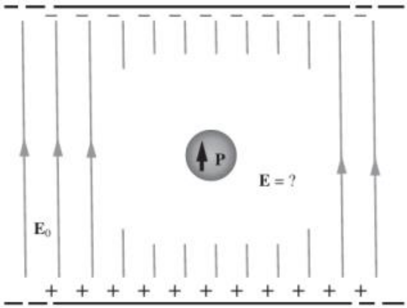
That is, we find more charge on each of the plates, for the same potential difference, plate area, and distance of separation.

Dielectric constants for some common substance where $\kappa > 1$ for all ordinary materials.

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

EXAMPLES:

1. Consider a dielectric sphere characterized by a dielectric constant κ into a homogeneous electric field $\vec{E}_0$. The dielectric sphere develops some polarization $\vec{P}$. Find the electric field inside the sphere.



ANSWER:

- Method I:

Inside the polarized sphere, the total field is

$\vec{E}' = \vec{E}_0 + \vec{E}_{\text{in}}$

Since

$\vec{P} = \epsilon_0(\kappa - 1)\vec{E}'$

and

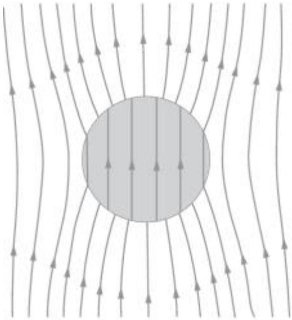
$$\vec{E}_{\text{in}} = -\frac{\vec{P}}{3\epsilon_0}$$

we obtain

$$\frac{\vec{P}}{\epsilon_0(\kappa - 1)} = \vec{E}_0 - \frac{\vec{P}}{3\epsilon_0} \Rightarrow \vec{P} = \frac{1}{\frac{1}{3} + \frac{1}{\kappa - 1}}\epsilon_0\vec{E}_0 = 3\left(\frac{\kappa - 1}{\kappa + 2}\right)\epsilon_0\vec{E}_0$$

The total field inside the sphere can be also expressed as

$$\vec{E}' = \vec{E}_0 - \frac{1}{3\epsilon_0}3\left(\frac{\kappa - 1}{\kappa + 2}\right)\epsilon_0\vec{E}_0 = \left(\frac{3}{\kappa + 2}\right)\vec{E}_0$$



- Method II:

  Suppose the electric field inside the sphere is equal to $\vec{E}_0$. Since the material is a linear dielectric the polarization is proportional to the total electric field:

  $\vec{P}_0 = \chi_e \epsilon_0 \vec{E}_0$

  A uniformly polarized sphere with polarization $\vec{P}_0$ produces an internal electric field equal to

  $$\vec{E}_1 = -\frac{\vec{P}_0}{3\epsilon_0} = -\frac{1}{3\epsilon_0}\chi_e \epsilon_0 \vec{E}_0 = -\frac{1}{3}\chi_e \vec{E}_0$$

  The electric field produced by the polarization of the sphere will therefore reduce the electric field inside the sphere. This change in the electric field will change the polarization of the sphere by

  $$\vec{P}_1 = \chi_e \epsilon_0 \vec{E}_1 = -\chi_e \epsilon_0 \frac{1}{3}\chi_e \vec{E}_0 = -\frac{1}{3}\epsilon_0 \chi_e^2 \vec{E}_0$$

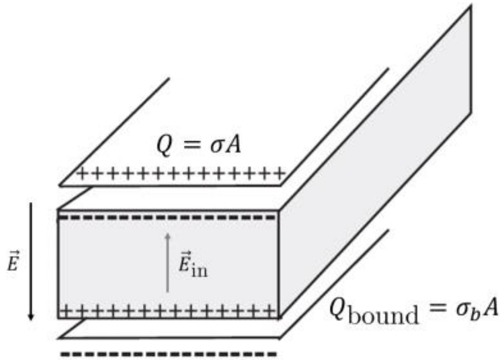
  This change in the polarization of the sphere will again change the electric field inside the sphere. This change of the electric field strength is equal to

  $$\vec{E}_2 = -\frac{\vec{P}_1}{3\epsilon_0} = -\frac{1}{3\epsilon_0}\left(-\frac{1}{3}\epsilon_0 \chi_e^2\right)\vec{E}_0 = \left(-\frac{1}{3}\chi_e\right)^2 \vec{E}_0$$

  This iterative process will continue indefinitely, and the final electric field will be equal to

  $$\vec{E}' = \sum_{n=0} \left(-\frac{1}{3}\chi_e\right)^n \vec{E}_0 = \frac{1}{1+\frac{1}{3}\chi_e}\vec{E}_0 = \frac{3}{3+\chi_e}\vec{E}_0 = \frac{3}{\kappa+2}\vec{E}_0$$

2. Find the bound charge density on the surface of the dielectric of a capacitor.



ANSWER:

In the interior of the dielectric, the applied field arising from the surface charge of the conductor, and the induced field arising from the oriented dipole,

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z}, \qquad \vec{E}_{\text{in}} = -\frac{\sigma_b}{\epsilon_0}\hat{z}$$

The total field in the dielectric is given by

$$\vec{E}' = \vec{E} + \vec{E}_{\text{in}}$$

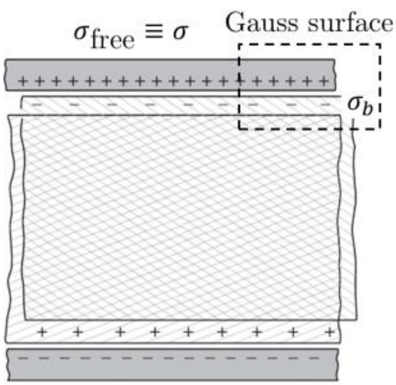
From equation (a), we have $\vec{E} = \kappa\vec{E}'$ and obtain

$$\vec{E}' = \frac{1}{\kappa}\vec{E} = \frac{\sigma}{\kappa\epsilon_0}\hat{z}$$

$$\frac{\sigma}{\kappa\epsilon_0} = \frac{\sigma}{\epsilon_0} - \frac{\sigma_b}{\epsilon_0} \Rightarrow \sigma_b = \left(1 - \frac{1}{\kappa}\right)\sigma = \left(\frac{\kappa - 1}{\kappa}\right)\sigma$$

B. GAUSS'S LAW WITH DIELECTRICS

(1) Choose a Gaussian surface in which contains both the charge on the conductor and that on the surface of the dielectric.



The total field in the dielectric is given by

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z} - \frac{\sigma_b}{\epsilon_0}\hat{z} = \left(\frac{\sigma}{\epsilon_0} - \left(\frac{\kappa - 1}{\kappa}\right)\frac{\sigma}{\epsilon_0}\right)\hat{z} = \frac{\sigma}{\kappa\epsilon_0}\hat{z}$$

Therefore, in terms of the free charge, we obtain

$$\kappa\epsilon_0\vec{E} = \sigma\hat{z}$$

Define the electric displacement field in the dielectric by

$$\vec{D} \equiv \kappa\epsilon_0\vec{E}$$

Gauss's law in the presence of dielectrics can then be written as

$$\oint_{\mathcal{S}} \vec{D} \cdot d\vec{a} = \oint_{\mathcal{S}} \sigma\hat{z} \cdot d\vec{a} = q_{\text{free}}$$

where q_{free} is the free charge contained inside the closed surface.

(2) Using Gauss's divergence theorem, we can obtain Gauss's law in differential form:

$$\oint_{\mathcal{S}} \vec{D} \cdot d\vec{a} = \int_{\mathcal{V}} \nabla \cdot \vec{D}\, d\tau = \int_{\mathcal{V}} \rho_{\text{free}}\, d\tau \Rightarrow \nabla \cdot \vec{D} = \rho_{\text{free}}$$

Since

$$\vec{D} = \kappa\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P}$$

we obtain

$$\nabla \cdot \left(\epsilon_0\vec{E} + \vec{P}\right) = \rho_{\text{free}} \Rightarrow \epsilon_0\nabla \cdot \vec{E} = -\nabla \cdot \vec{P} + \rho_{\text{free}}$$

Since

$\rho_b = -\nabla \cdot \vec{P}$ ····· the bound charge within the dielectric

thus, we have

$$\epsilon_0\nabla \cdot \vec{E} = \rho_b + \rho_{\text{free}} = \rho$$

In any system whatsoever, the fundamental relation (namely Gauss's law) between the total electric field $\vec{E}$ within the dielectric and the total charge density ρ remains valid.

(3) If additionally we are dealing with a linear dielectric, then

$$\vec{D} = \kappa\epsilon_0\vec{E} = \epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E}$$

where $\epsilon = \kappa\epsilon_0$ is known as the permittivity of the dielectric, and ϵ_0 is called the permittivity of free space.

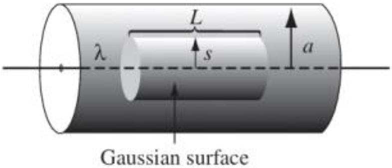
Gauss's law becomes

$$\nabla \cdot \vec{D} = \epsilon\nabla \cdot \vec{E} = \rho_{\text{free}} \Rightarrow \nabla \cdot \vec{E} = \frac{\rho_{\text{free}}}{\epsilon}$$

$$\oint_{\mathcal{S}} \vec{D} \cdot d\vec{a} = \epsilon\oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = q_{\text{free}} \Rightarrow \oint_{\mathcal{S}} \vec{E} \cdot d\vec{a} = \frac{q_{\text{free}}}{\epsilon}$$

EXAMPLES:

1. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement.



ANSWER:

Using Gauss's law, we have

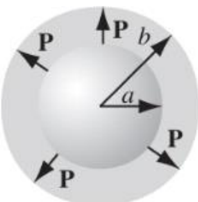
$$\oint_S \vec{D} \cdot d\vec{a} = q_{\text{free}} \Rightarrow D \cdot 2\pi s L = \lambda L \Rightarrow \vec{D} = \frac{\lambda}{2\pi s}\hat{r}$$

Outside the rubber, $\vec{P} = 0$, the field is

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} - \vec{P} = \frac{\vec{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{r}$$

Inside the rubber, since we do not know $\vec{P}$, the field cannot be determined.

2. A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization $\vec{P}(r) = \frac{k}{r}\hat{r}$, where k is a constant and r is the distance from the center. Find the electric field.



ANSWER:

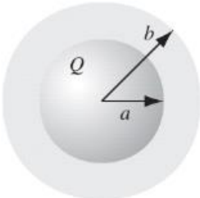
Since $q_{\text{free}} = 0$

$$\oint_S \vec{D} \cdot d\vec{a} = q_{\text{free}} = 0 \Rightarrow \vec{D} = 0 \text{ everywhere}$$

Since $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = 0$, so we have

$$\vec{E} = -\frac{\vec{P}}{\epsilon_0} = \begin{cases} 0 & , \quad r < a \text{ or } r > b \\ -\dfrac{k}{\epsilon_0 r}\hat{r} & , \quad a < r < b \end{cases}$$

3. A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center.



ANSWER:

Outside the metal sphere:

$$\oint_S \vec{D} \cdot d\vec{a} = q_{\text{free}} = Q \Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}, \qquad \text{for all points } r > a$$

For $a < r < b$:

$$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

For $r > b$:

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

Inside the metal sphere:

$$\vec{E} = 0, \ \vec{D} = \epsilon\vec{E} = 0$$

The potential at the center is therefore

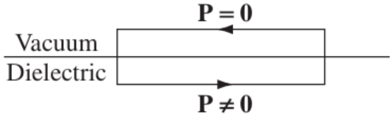
$$\varphi = -\int_{\infty}^{0} \vec{E} \cdot d\vec{s}$$
$$= -\int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{b}^{a} \frac{Q}{4\pi\epsilon r^2} dr - \int_{a}^{0} 0 \, dr$$
$$= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_0}\left(\frac{1}{b} - \frac{1}{\infty}\right) + \frac{1}{\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$
$$= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right]$$

C. BOUNDARY-VALUE PROBLEMS WITH DIELECTRICS

(1) According to Helmholtz theorem, the electrostatic field $\vec{E}$ is uniquely determined by

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ and } \nabla \times \vec{E} = 0$$

Since the curl of $\vec{D}$ is not always zero:



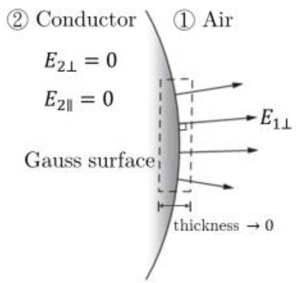
$$\nabla \times \vec{D} = \nabla \times \left(\epsilon_0 \vec{E} + \vec{P}\right) = \nabla \times \vec{P} \stackrel{?}{=} 0$$

$\vec{D}$ cannot be uniquely determined by the free charge only as

$$\nabla \cdot \vec{D} = \rho_{\text{free}}$$

Thus, we need boundary conditions on $\vec{D}$ at various dielectric surfaces.

(2) Recall that for a charged conductor with surface charge density σ. We choose a Gaussian surface for an extreme small area $d\vec{a}$ and let the thickness go to zero to avoid the parallel components of $\vec{E}$ through the Gaussian surface.

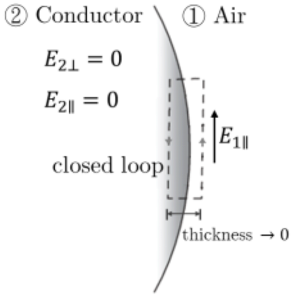


Thus, we obtain

$$\oint_S \vec{E} \cdot d\vec{a} = \underbrace{E_{1\perp} da}_{\text{Air}} - \underbrace{E_{2\perp}}_{\substack{=0 \\ \text{conductor}}} da = \frac{\sigma da}{\epsilon_0} \Rightarrow \underbrace{E_{1\perp}}_{\text{Air}} - \underbrace{E_{2\perp}}_{\text{conductor}} = \frac{\sigma}{\epsilon_0}$$

$E_\perp$ is discontinuous across the surface of the conductor by an amount σ/ϵ_0.

For if we use Stokes' theorem and let the width of the closed loop go to zero.

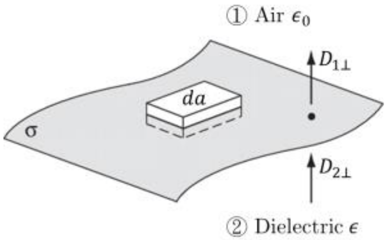


Thus, we obtain

$$\oint_{\mathcal{C}} \vec{E} \cdot d\vec{s} = \underbrace{E_{1\parallel} ds}_{\text{Air}} - \underbrace{E_{2\parallel}}_{\substack{=0 \\ \text{conductor}}} ds = 0 \Rightarrow \underbrace{E_{1\parallel}}_{\text{Air}} - \underbrace{E_{2\parallel}}_{\text{conductor}} = 0$$

$E_{\parallel}$ is continuous across the surface of the conductor.

(3) Now back to the electric displacement field $\vec{D}$, we choose a Gaussian surface for a very tiny area $d\vec{a}$ and let the thickness go to zero.



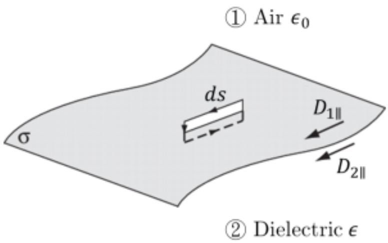
Thus, we obtain

$$\oint_{\mathcal{S}} \vec{D} \cdot d\vec{a} = \underbrace{D_{1\perp} da}_{\text{Air}} - \underbrace{D_{2\perp} da}_{\text{dielectric}} = \sigma_{\text{free}} da \Rightarrow \underbrace{D_{1\perp}}_{\text{media ①}} - \underbrace{D_{2\perp}}_{\text{media ②}} = \sigma_{\text{free}}$$

$D_{\perp}$ is discontinuous across the surface of the dielectric. For linear dielectrics $\vec{D} = \epsilon\vec{E}$, we have

$$\underbrace{\epsilon_1 E_{1\perp}}_{\text{media ①}} - \underbrace{\epsilon_2 E_{2\perp}}_{\text{media ②}} = \sigma_{\text{free}}$$

We then can choose a closed loop such that the width goes to zero as



Since

$$\oint_{\mathcal{C}} \vec{D} \cdot d\vec{s} = \int_{\mathcal{S}} \left(\nabla \times \vec{D}\right) \cdot d\vec{a} = \int_{\mathcal{S}} \left(\nabla \times \vec{P}\right) \cdot d\vec{a} = \oint_{\mathcal{C}} \vec{P} \cdot d\vec{s}$$

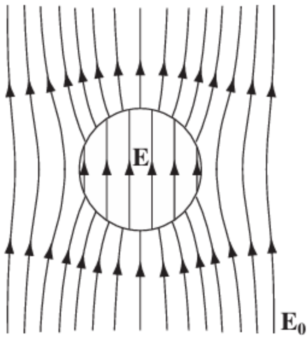
we obtain

$$\underbrace{D_{1\parallel} ds}_{\text{Air}} - \underbrace{D_{2\parallel} ds}_{\text{dielectric}} = \underbrace{P_{1\parallel} ds}_{\text{Air}} - \underbrace{P_{2\parallel} ds}_{\text{dielectric}} \Rightarrow D_{1\parallel} - D_{2\parallel} = P_{1\parallel} - P_{2\parallel}$$

$D_{\parallel}$ is also discontinuous across the surface of the dielectric.

EXAMPLES:

1. A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. Find the electric field inside the sphere.



ANSWER:

The solution of Laplace's equation for φ is given by the method of separation of variables as

$$\varphi = \begin{cases} \displaystyle\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & , \quad \text{inside the sphere} \\ \displaystyle -E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \quad \text{outside the sphere} \end{cases}$$

Since there are two unknown variables, we need boundary conditions at $r = R$ to determine the values.

$\varphi_{\text{out}}(R) = \varphi_{\text{in}}(R)$ ····· (a)

$D_{1\perp}(R) - D_{2\perp}(R) = \sigma_f$ ····· (b)

Since there is no free charge at the surface, thus, we have $\sigma_f = 0$ and obtain

$$D_{1\perp} - D_{2\perp} = 0 \Rightarrow \epsilon_0 E_{1\perp} - \epsilon E_{2\perp} = \epsilon_0 \left.\frac{\partial \varphi_{\text{out}}}{\partial r}\right|_R - \epsilon \left.\frac{\partial \varphi_{\text{in}}}{\partial r}\right|_R = 0 \text{ ····· (c)}$$

From equation (a), we obtain

$$-E_0 R\cos\theta + \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) = \sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)$$

Only $l = 1$ survived. Thus, we obtain

$$-E_0 R + \frac{B_1}{R^2} = A_1 R$$

From equation (c), we obtain

$$-\epsilon_0 E_0\cos\theta - \epsilon_0\sum_{l=0}^{\infty}\frac{(l+1)B_l}{R^{l+2}}P_l(\cos\theta) = \epsilon\sum_{l=0}^{\infty}lA_l R^{l-1}P_l(\cos\theta)$$

Only $l = 1$ survived. Thus, we obtain

$$-\epsilon_0 E_0 - \frac{2B_1}{R^3} = \epsilon A_1$$

Together, we can solve the set of equations for A_1 and B_1,

$$A_1 = -\frac{3}{\epsilon/\epsilon_0 + 2}E_0, \qquad B_1 = \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}R^3 E_0$$

The potential inside the sphere is

$$\varphi_{\text{in}} = A_1 r^1 P_1(\cos\theta) = -\frac{3}{\epsilon/\epsilon_0 + 2}E_0 r\cos\theta = -\frac{3}{\kappa + 2}\vec{E}_0 \cdot z\hat{z}$$

The total field inside the sphere is

$$\vec{E}_{\text{in}} = -\nabla\varphi = -\frac{\partial\varphi}{\partial z}\hat{z} = \left(\frac{3}{\kappa + 2}\right)\vec{E}_0$$

10-4 Electrostatic Energy and Force

A. ENERGY STORED IN THE DIELECTRICS

(1) Recall that the energy stored in the conductor

$$W = \frac{1}{2}\int \rho\varphi\, d\tau = \frac{\epsilon_0}{2}\int E^2\, d\tau = \frac{1}{2}CV^2$$

Since the first term contains the source charge density ρ and the potential function V, it is a general formula for the energy stored in any electrostatic system, for example, conductors or dielectrics.

(2) Consider the energy stored in the dielectrics

Since

$$\nabla \cdot \vec{D} = \rho$$

we can express the energy in terms of $\vec{D}$.

$$W = \frac{1}{2}\int \rho V\, d\tau = \frac{1}{2}\int \left(\nabla \cdot \vec{D}\right) V\, d\tau$$

Since

$$\nabla \cdot \left(V\vec{D}\right) = V\left(\nabla \cdot \vec{D}\right) + \vec{D} \cdot \nabla V$$

thus, we obtain

$$W = \frac{1}{2}\int_{\mathcal{V}} \nabla \cdot \left(V\vec{D}\right) d\tau - \frac{1}{2}\int_{\mathcal{V}} \left(\vec{D} \cdot \nabla\right) V\, d\tau$$

$$= \frac{1}{2}\oint_{\mathcal{S}} V\vec{D} \cdot d\vec{a} - \frac{1}{2}\int_{\mathcal{V}} \left(\vec{D} \cdot \nabla\right) V\, d\tau$$

As we choose a very large sphere with radius r, φ falls of like $1/r$ and $\vec{D}$ falls off like $1/r^2$. The area of the closed surface increases like r^2. Hence the surface integral decreases as fast as $1/r$ and will vanish as $r \to \infty$. Thus, we obtain

$$W = -\frac{1}{2}\int_{\mathcal{V}} \left(\vec{D} \cdot \nabla\right) V\, d\tau = \frac{1}{2}\int_{\mathcal{V}} \vec{D} \cdot \vec{E}\, d\tau$$

Using the relation $\vec{D} = \epsilon\vec{E}$, we obtain

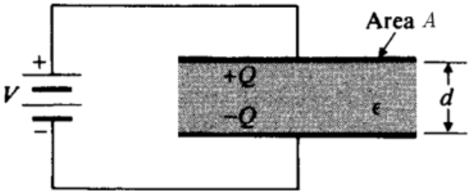
$$W = \frac{1}{2}\int \epsilon\vec{E} \cdot \vec{E}\, d\tau = \frac{\epsilon}{2}\int E^2\, d\tau = \frac{1}{2\epsilon}\int D^2\, d\tau$$

(3) Thus, we define an electrostatic energy density as

$$w = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{\epsilon}{2}E^2 = \frac{1}{2\epsilon}D^2$$

EXAMPLES:

1. In a parallel-plate capacitor of area A and separation d is charged to a voltage V. The permittivity of the dielectric is ϵ. Find the stored electricstatic energy.



ANSWER:

Suppose that the fringing of the field at the edge is neglected, the electric field in the dielectric is uniform,

$$E = \frac{V}{d}$$

Thus, we have

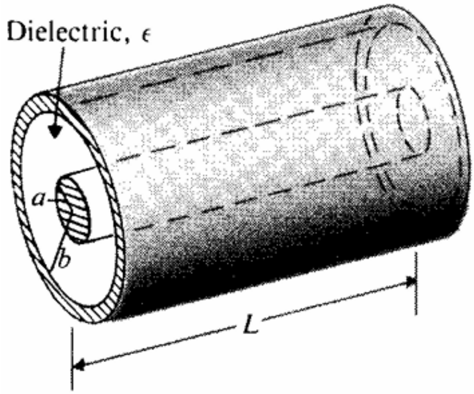
$$W = \frac{\epsilon}{2}\int E^2\, d\tau = \frac{\epsilon}{2}\int \left(\frac{V}{d}\right)^2 d\tau = \frac{\epsilon}{2}\left(\frac{V}{d}\right)^2 (Ad) = \frac{1}{2}\left(\frac{\epsilon A}{d}\right)V^2$$

Since $C = \epsilon A/d$ and $Q = CV$, the energy can be expressed as

$$W = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}$$

This formula hold true for any parallel-plate capacitor or two-conductor capacitor.

2. A cylindrical capacitor consists of an inner conductor of radius a and outer conductor whose inner radius is b. The space between the conductors is filled with a dielectric of permittivity ϵ, and the length of the capacitor is L. Determine the capacitance.



ANSWER:

- Method I:

  Using Gauss's law, we have

  $\vec{E} = \frac{q}{2\pi\epsilon L r}\hat{r}$

  Suppose that the fringing of the field at the edge is neglected.

  The potential difference between the inner and outer conductor is

  $\varphi = -\int_b^a \vec{E}\cdot d\vec{s} = -\int_b^a \frac{q}{2\pi\epsilon L r}\hat{r}\cdot dr\hat{r} = \frac{q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)$

  The capacitance is

  $C = \frac{q}{\varphi} = \frac{q}{\frac{q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$

- Method II:

  The energy stored in the dielectric region is

  $W = \frac{\epsilon}{2}\int E^2\,d\tau = \frac{\epsilon}{2}\int_a^b \left(\frac{q}{2\pi\epsilon L r}\right)^2 (2\pi r L dr) = \frac{q^2}{4\pi\epsilon L}\ln\left(\frac{b}{a}\right)$

  Since

  $W = \frac{q^2}{2C} = \frac{q^2}{4\pi\epsilon L}\ln\left(\frac{b}{a}\right) \Rightarrow C = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$

B. FORCE ON THE DIELECTRICS

(1) System of bodies with fixed charges

Image that the electric forces have displaced one of the bodies by a differential distance $d\vec{s}$ (a virtual displacement). The work done by the system is

$$dW = \vec{F}_q \cdot d\vec{s}$$

The work done is the expense of the stored energy

$$dW = -dW_e$$

Since

$$dW_e = \nabla W_e \cdot d\vec{s}$$

thus, we have

$$\vec{F}_q = -\nabla W_e$$

(2) System of bodies with fixed potentials

If a charge dq_k is added to the k-th matter that is maintained at potential φ_k, the work done by the sources is $\varphi_k dq_k$. The total work done by the sources to the system is

$$dW_s = \sum_k \varphi_k \, dq_k$$

The work done by the system is

$$dW = \vec{F}_\varphi \cdot d\vec{s}$$

The potential energy of the distributed charges is

$$dW_e = \frac{1}{2}\sum_k \varphi_k \, dq_k = \frac{1}{2} dW_s$$

Conservation of energy demands that

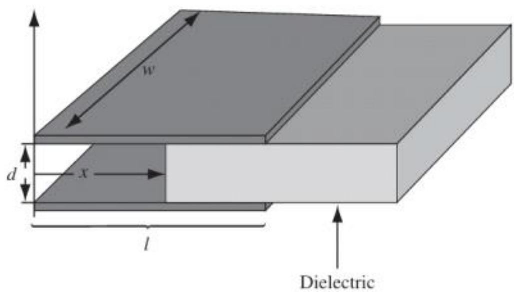
$$dW_s = dW + dW_e \Rightarrow 2dW_e = \vec{F}_\varphi \cdot d\vec{s} + dW_e \Rightarrow \vec{F}_\varphi \cdot d\vec{s} = \nabla W_e \cdot d\vec{s}$$

Thus, we obtain

$$\vec{F}_\varphi = \nabla W_e$$

EXAMPLES:

1. Consider a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor. The fringing field around the edges pulls the dielectric into the capacitor. Determine the force on the slab.



ANSWER:

- Fixed charges:

$$W_e = \frac{1}{2}\frac{q^2}{C}$$

$$F_q = -\frac{dW_e}{dx} = \frac{1}{2}\frac{q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

Since

$$C = \frac{\epsilon_0 w}{d}(\kappa l - \chi x)$$

$$F_q = \frac{1}{2}V^2\left(-\frac{\epsilon_0 w}{d}\chi\right) = -\frac{\epsilon_0 \chi w}{2d}V^2$$

- Fixed potentials:

$$W_e = \frac{1}{2}CV^2$$

$$F_q = \frac{1}{2}V^2\frac{dC}{dx} = \frac{1}{2}V^2\left(-\frac{\epsilon_0 w}{d}\chi\right) = -\frac{\epsilon_0 \chi w}{2d}V^2$$